About construction of orthogonal wavelets with compact support and with scaling coefficient N

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In paper [1] with using of the Cuntz algebra representation some methods of construction of wavelets with scaling coefficient $N \geq 2$ are considered. In paper [2] it is shown, a construction of wavelets at the prescribed scaling function $\varphi(x)$. In this paper a simple method of construction of scaling function $\varphi(x)$ and orthogonal wavelets with the compact support for any natural coefficient of scaling $N \geq 2$ is given. Examples of construction of wavelets for coefficients of scaling $N = 2$ and $N = 3$ are produced.

1. Scaling functions and wavelets. Let $N \geq 2$ is an integer, $\mathbb{Z}$ is set of all integers and $L^2(\mathbb{R})$ is Hilbert space of square integrable functions.

Definition 1. *Function $\varphi(x) \in L^2(\mathbb{R})$ is called N-scaling, if it can be represented as*

$$\varphi(x) = \sqrt{N} \sum_{n \in \mathbb{Z}} h_n \varphi(Nx - n), \tag{1}$$

where coefficients h_n, $n \in \mathbb{Z}$ satisfy to condition $\sum_n |h_n|^2 < \infty$. The relationship (1) is called the N-scale equation (refinement equation). The set $\{h_n\}$ of coefficients of expansion in the equation (1) is called the scaling filter.

Note 1. If N-scaling function $\varphi(x)$ has the compact support of length L, then the sum in equation (1) is finite, contained at most $L(N-1)+1$ components.

The Fourier transform of N-scale equation is

$$\widehat{\varphi}(\omega) = H_0\left(\frac{\omega}{N}\right) \widehat{\varphi}\left(\frac{\omega}{N}\right), \tag{2}$$

where

$$H_0(\omega) = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} h_n e^{-in\omega}. \tag{3}$$

The function $H_0(\omega)$ is called *frequency function* of scaling function $\varphi(x)$.

In the orthogonal case translations of scaling function $\varphi(x-n)$, $n \in \mathbb{Z}$ form orthonormal basis of the subspace V_0 in $L^2(\mathbb{R})$, and translations $\varphi_{1,n}(x) = \sqrt{N}\varphi(Nx-n)$, $n \in \mathbb{Z}$ on $1/N$, form orthonormal basis of the subspace V_1 in $L^2(\mathbb{R})$. Thus $V_0 \subset V_1$. In the orthogonal case to the scaling function $\varphi(x)$ corresponds $N-1$ wavelets-functions $\psi^1(x)$..., $\psi^{N-1}(x)$, for each of which translations $\psi^k_{0,n}(x) = \psi^k(x-n)$, $n \in \mathbb{Z}$ form orthonormal basis of subspaces W^k_0 in $L^2(\mathbb{R})$, and expansion in the direct sum of orthogonal subspaces $V_1 = V_0 \oplus W^1_0 \oplus \cdots \oplus W^{N-1}_0$ be valid.

Wavelets $\psi^1(x)$..., $\psi^{N-1}(x)$ form orthonormal basis $L^2(\mathbb{R})$:

$$\{\psi^k_{j,n}(x) = \sqrt{N^j}\psi^k(N^j x - n),\ j,n \in \mathbb{Z},\ k = 1,2,\ldots,N-1\}.$$

As wavelets $\psi^1(x)$..., $\psi^{N-1}(x)$ belong to space V_1 they are decomposed on basis of this space,

$$\psi^k(x) = \sqrt{N}\sum_{n\in\mathbb{Z}} g^k_n \varphi(Nx - n). \tag{4}$$

The coefficients $\{g^k_n\}$ is called *filters of wavelets* $\psi^k(x)$, $k = 1,2,\ldots,N-1$. Let

$$H_k(\omega) = \frac{1}{\sqrt{N}}\sum_{n\in\mathbb{Z}} g^k_n e^{-in\omega} \tag{5}$$

– the frequency functions corresponding to wavelets $\psi^1(x)$..., $\psi^{N-1}(x)$. The Fourier transform of equalities (4) is

$$\widehat{\psi}^k(\omega) = H_k\left(\frac{\omega}{N}\right)\widehat{\varphi}\left(\frac{\omega}{N}\right).$$

For the frequency functions $H_k(\omega)$ the following matrix is unitary [2], [3],

$$H(z) = \begin{pmatrix} H_0(z) & H_0(\rho z) & \ldots & H_0(\rho^{N-1}z) \\ H_1(z) & H_1(\rho z) & \ldots & H_1(\rho^{N-1}z) \\ \ldots & \ldots & \ldots & \ldots \\ H_{N-1}(z) & H_{N-1}(\rho z) & \ldots & H_{N-1}(\rho^{N-1}z) \end{pmatrix}, \tag{6}$$

where $z = e^{-i\omega}$ and $\rho = e^{-i2\pi/N}$. The matrix (6) has special view. It is possible to avoid of this special view of the matrix $H(z)$ with Fourier transform on cyclic group $\mathbb{Z}/N\mathbb{Z} = \{1, \rho, \rho^2, \ldots, \rho^{N-1}\}$ [2]. We shall define

$$A_{k,j}(w) = \frac{1}{\sqrt{N}}\sum_{z^N = w} z^{-j}H_k(z). \tag{7}$$

It is easy to see, that the sum on the right depends from $w = z^N$. Also transformation (7) accurate within coefficient $\sqrt{N}$ is sample of elements with degrees z^{kN} in polynomials $H_k(z)$, $z^{-1}H_k(z)$..., $z^{-N+1}H_k(z)$. Inverse transformation is defined by the formula [2]

$$H_k(z) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} z^j A_{k,j}(z^N). \tag{8}$$

From last relation we shall obtained the following matrix equality:

$$H(z) = \frac{1}{\sqrt{N}}A(z^N)\begin{pmatrix} 1 & 1 & \ldots & 1 \\ z & \rho z & \ldots & \rho^{N-1}z \\ \ldots & \ldots & \ldots & \ldots \\ z^{N-1} & \rho^{N-1}z^{N-1} & \ldots & \rho^{((N-1)^2)}z^{N-1} \end{pmatrix} = A(z^N)R(z). \tag{9}$$

In this expression the matrix $A(z^N)$ is already arbitrary unitary matrix with polynomial elements. Now specificity of the matrix $H(z)$ go to the matrix

$$R(z) = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & \ldots & 1 \\ z & \rho z & \ldots & \rho^{N-1}z \\ \ldots & \ldots & \ldots & \ldots \\ z^{N-1} & \rho^{N-1}z^{N-1} & \ldots & \rho^{(N-1)^2}z^{N-1} \end{pmatrix}. \tag{10}$$

Let's mark, that the matrix $R(z)$ is unitary on the unit circle $z = e^{-i\omega}$.

Specifying the polyphase matrix $A(w)$, we can construct the matrix of frequency functions $H(z)$ by the formula (8) and, together with it, frequency functions of wavelets $H_1(z), \ldots, H_{N-1}(z)$, hence, and wavelets $\psi^1(x) \ldots, \psi^{N-1}(x)$.

In work [2] the scheme of construction of the polyphase matrix $A(z^N)$ is given in the supposition, that polynomial frequency function $H_0(z)$ is prescribed. Then it is possible to consider, that the first row of the matrix $A_{0j}(z^N)$ is known,

$$A_{0,j}(w) = \frac{1}{\sqrt{N}} \sum_{z^N = w} z^{-j} H_0(z), \tag{11}$$

and it is necessary to construct remaining row of the matrix $A(w)$.

In the given work we shall give the simple scheme of construction of the unitary matrix $A(w)$ which elements are polynomials with real coefficients. It allows to define both the scaling function $\varphi(x)$ with compact support and with scaling coefficient $N > 2$, and orthogonal wavelets $\psi^1(x) \ldots, \psi^{N-1}(x)$.

2. Scheme of wavelets construction. From above constructions and methods of work [2] follows that orthogonal systems of wavelets can be determine by the unitary matrix $A(w)$ with polynomial elements with using of the formula $H(z) = A(z^N)R(z)$, where $R(z)$ – the special matrix (10). We shall give the simple method of construction enough big set of unitary matrixes $A(w)$ with polynomial elements. It will allow to obtain both the N-scaling function with the compact support, and orthogonal wavelets.

Let's choose any orthogonal matrix $A_0 = \{a_{ij},\ i, j = 0, 1, \ldots, N-1\}$ of the order $N \geq 2$. We shall multiply it on the diagonal unitary matrix $D_k(w) = \mathrm{diag}(w^{k_0}, w^{k_1}, \ldots, w^{k_{N-1}})$, where $k = (k_0, k_1, \ldots, k_{N-1})$ is set of integers and $|w| = 1$, and then – on the orthogonal matrix $B_0 = \{b_{ij},\ i, j = 0, 1, \ldots, N-1\}$. In outcome we shall obtain unitary matrix

$$A(w) = A_0 D_k(w) B_0, \tag{12}$$

which elements, $A_{ij} = \sum_{s=0}^{N-1} a_{is} b_{sj} w^{k_s}$, are polynomials on the variable w with real coefficients.

Now we shall substitute $w = z^N$, where $z = e^{-i\omega}$. We shall obtain the unitary matrix $A(z^N)$ with polynomial elements and real coefficients. We shall multiply it on the unitary matrix $R(z)$. Then we shall obtain the unitary matrix $H(z)$ of frequency polynomial functions $H_0(z)$, $H_1(z)$..., $H_{N-1}(z)$ of orthogonal system of wavelets $\varphi(x)$, $\psi^1(x) \ldots, \psi^{N-1}(x)$, where the first function $\varphi(x)$ is scaling, and remaining – wavelets. Thus,

$$H(z) = \begin{pmatrix} H_0(z) & H_0(\rho z) & \ldots & H_0(\rho^{N-1} z) \\ H_1(z) & H_1(\rho z) & \ldots & H_1(\rho^{N-1} z) \\ \ldots & \ldots & \ldots & \ldots \\ H_{N-1}(z) & H_{N-1}(\rho z) & \ldots & H_{N-1}(\rho^{N-1} z) \end{pmatrix} = A_0 D_k(z^N) B_0 R(z). \tag{13}$$

From (13) follows the expression for frequency functions:

$$H_k(z) = \frac{1}{\sqrt{N}} \sum_{s,j=0}^{N-1} a_{ks} b_{sj} z^j z^{N k_s},\ \ k = 0, 1, \ldots, N-1. \tag{14}$$

In order to the obtained the functions $H_k(z)$ would be frequency functions of orthogonal wavelets, it is necessary, that the sum of coefficients for $H_0(z)$ would be equal to unit, and the sums of coefficients for remaining functions $H_1(z)$..., $H_{N-1}(z)$ would be equal to zero:

$$\frac{1}{\sqrt{N}} \sum_{s,j=0}^{N-1} a_{0s} b_{sj} = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} a_{0s} \sum_{j=0}^{N-1} b_{sj} = 1,$$

$$\frac{1}{\sqrt{N}}\sum_{s,j=0}^{N-1} a_{ks}b_{sj} = \frac{1}{\sqrt{N}}\sum_{s=0}^{N-1} a_{ks}\sum_{j=0}^{N-1} b_{sj} = 0, \qquad k = 0,1,\ldots,N-1.$$

These equalities can be represented in the matrix view:

$$\left(\begin{array}{cccc} a_{00} & a_{01} & \ldots & a_{0,N-1} \\ a_{10} & a_{11} & \ldots & a_{1,N-1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{N-1,0} & a_{N-1,1} & \ldots & a_{N-1,,N-1} \end{array}\right)\left(\begin{array}{c} b_{00}+\cdots+b_{0,N-1} \\ b_{10}+\cdots+b_{1,N-1} \\ \ldots \\ b_{N-1,0}+\cdots+b_{N-1,,N-1} \end{array}\right) = \left(\begin{array}{c} \sqrt{N} \\ 0 \\ \ldots \\ 0 \end{array}\right). \qquad (15)$$

Choosing various orthogonal matrixes A_0 and B_0, which satisfy the equality (15), we obtain various frequency functions of wavelets (14).

For construction enough simple class of orthogonal wavelets with the compact support and scaling coefficient $N > 2$, we shall take as an orthogonal matrix A_0 the following matrix:

$$A_0 = \left(\begin{array}{ccccc} 1/\sqrt{N} & 1/\sqrt{N} & 1/\sqrt{N} & \ldots & 1/\sqrt{N} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \ldots & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1/\sqrt{N(N-1)} & 1/\sqrt{N(N-1)} & 1/\sqrt{N(N-1)} & \ldots & -(N-1)/\sqrt{N(N-1)} \end{array}\right).$$

The matrix A_0 transform vector of units $e = (1,1,\ldots,1)$ to the vector $\sqrt{N}e_0 = (\sqrt{N},0,\ldots,0)$, $A_0 e = \sqrt{N}e_0$. Then from equality (15) follows, that elements of the orthogonal matrix B_0 should satisfy to the following system of equations:

$$\left\{\begin{array}{lcl} b_{00}+b_{01}+\ldots+b_{0,N-1} & = & 1 \\ b_{10}+b_{11}+\ldots+b_{1,N-1} & = & 1 \\ \ldots & \ldots & \ldots \\ b_{N-1,0}+b_{N-1,1}+\ldots+b_{N-1,N-1} & = & 1 \end{array}\right. . \qquad (16)$$

The solution of this system will be any set of orthonormal vectors (rows) which coordinates satisfy to the equation of the plane $x_0+x_1+\ldots+x_{N-1} = 1$ in $\mathbb{R}^N$. It is obvious, that coordinates of basis vectors $e_0 = (1,0,\ldots,0)$, $e_1 = (0,1,0,\ldots,0)$...,$e_{N-1} = (0,\ldots,0,1)$ satisfy to this equation. The given solution corresponds to the identity matrix B_0. Any other solution can be obtained by rotation of the basis solution $e_0, e_1, \ldots, e_{N-1}$ around of vector $e = e_0+e_1+\cdots+e_{N-1}$, i.e. in the plane $x_0+x_1+\ldots+x_{N-1} = 1$. We shall find these solutions. We shall take rotation around of axis Ox_0:

$$M = \left(\begin{array}{ccccc} 1 & 0 & 0 & \ldots & 0 \\ 0 & m_1^1 & m_2^1 & \ldots & m_{N-1}^1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & m_1^{N-1} & m_2^{N-1} & \ldots & m_{N-1}^{N-1} \end{array}\right). \qquad (17)$$

As $A_0 e = \sqrt{N}e_0$, then rotation around of axis e is given by the matrix $M_e = A_0^{-1}MA_0$. Then rows of the matrix B_0 will consist of coordinates of vectors-columns which are obtained from $e_0, e_1, \ldots, e_{N-1}$ by action on them matrix M_e. Therefore the matrix B_0 is transposed to M_e. Then

$$H_M(z) = A_0 D_k(z^N) M_e^T R(z) = A_0 D_k(z^N) A_0^T M^T A_0 R(z), \qquad (18)$$

where M – any orthogonal matrix of view (17) and $D_k(w) = \mathrm{diag}(w^{k_0}, w^{k_1}, \ldots, w^{k_{N-1}})$.

The formula (18) gives the direct method of construction the big family of frequency functions $H_0(z)$, $H_1(z)$..., $H_{N-1}(z)$ and orthogonal wavelets with the compact support $\varphi(x)$, $\psi^1(x)\ldots,\psi^{N-1}(x)$. Wavelets of the family depend of the orthogonal matrix M of view (17) and of the vector of degrees $k=(k_0,k_1,\ldots...,k_{N-1})$ which it is possible to set arbitrarily.

3. Construction of orthogonal wavelets with compact support for $N=2$. In the given section we shall show by the example of scale $N=2$ effectiveness of the wavelets construction scheme explained above. Though the matrix $D_k(w)$ can be anyone, we shall take for example the diagonal matrix $D_1(w)=\operatorname{diag}\{1,w\}$, $|w|=1$. In case $N=2$ orthogonal matrixes A_0 and B_0 can be in the general view:

$$A_0=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},\qquad B_0=\begin{pmatrix}\cos u & \sin u\\ -\sin u & \cos u\end{pmatrix}.$$

Then

$$H(z)=\frac{1}{\sqrt{2}}\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^2\end{pmatrix}\begin{pmatrix}\cos u & \sin u\\ -\sin u & \cos u\end{pmatrix}\begin{pmatrix}1 & 1\\ z & \rho z\end{pmatrix}.$$

Frequency functions are

$$H_0(z)=\frac{1}{\sqrt{2}}\left(\cos t\cos u+(\cos t\sin u)z-(\sin t\sin u)z^2+(\sin t\cos u)z^3\right),\tag{19}$$

$$H_1(z)=\frac{1}{\sqrt{2}}\left(-\sin t\cos u-(\sin t\sin u)z-(\cos t\sin u)z^2+(\cos t\cos u)z^3\right),\tag{20}$$

The sum of coefficients of frequency function $H_0(z)$ should be equal to unit, and the sum of coefficients of frequency function $H_1(z)$ should be equal to zero. The system (15) becomes:

$$\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}\cos u+\sin u\\ \cos u-\sin u\end{pmatrix}=\begin{pmatrix}\sqrt{2}\\ 0\end{pmatrix},$$

$$\left\{\begin{array}{lcl}\cos u+\sin u & = & \sqrt{2}\cos t\\ \cos u-\sin u & = & \sqrt{2}\sin t\end{array}\right. .$$

Solving last system, we obtain, $u=\pi/4-t$.

Thus, we have constructed the family of frequency functions of the wavelets specified by formulas (19), (20) in which $u=\pi/4-t$. After elimination of the variable u, we obtain::

$$H_0(z)=\frac{1}{4}\left(1+\cos 2t+\sin 2t+(1+\cos 2t-\sin 2t)z+\right.$$

$$\left.+(1-\cos 2t-\sin 2t)z^2+(1-\cos 2t+\sin 2t)z^3\right),\tag{21}$$

$$H_1(z)=\frac{1}{4}\left(-1+\cos 2t-\sin 2t+(1-\cos 2t-\sin 2t)z+\right.$$

$$\left.+(-1-\cos 2t+\sin 2t)z^2+(1+\cos 2t+\sin 2t)z^3\right).\tag{22}$$

The given frequency functions $H_0(z)$ and $H_1(z)$ coincide with the same, but obtained other methods in work [1]. Various wavelets of Haar, Daubechies wavelets and their analogs include into this family. In the following section some examples are given.

Choosing other matrix $D_k(z^N)$, similarly we can construct other orthogonal wavelets with other support length.

4. Examples of scaling functions and wavelets for $N=2$. We shall calculate values of coefficients of the obtained frequency functions (19), (20) for various parameters

t and u and we shall find corresponding filters and wavelets $\varphi(x)$ and $\psi(x)$. From formulas (21), (22) follows what enough to take parameter values t on interval of length π. We shall consider the following parameter values t: $0, \pm\pi/12, \pm\pi/6, \pm\pi/4, \pm\pi/3, \pm5\pi/12, \pi/2$.

4.1 Parameter values $t = 0$, $u = \pi/4$. Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \qquad g_1 = \frac{1}{\sqrt{2}}(0, 0, -1, 1).$$

We have obtained wavelets of Haar with the support on unit interval. Refinement equations: $\varphi(x) = \varphi(2x) + \varphi(2x-1)$ and $\psi(x) = -\psi(2x-2) + \psi(2x-3)$.

4.2. Parameter values $t = \pi/4$, $u = 0$. Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \qquad g_1 = \frac{1}{\sqrt{2}}(-1, 0, 0, 1).$$

We have obtained wavelets of Haar with the support on interval $[0, 3]$. Refinement equations: $\varphi(x) = \varphi(2x) + \varphi(2x-3)$ and $\psi(x) = -\psi(2x) + \psi(2x-3)$.

4.3. Parameter values $t = \pi/2$, $u = \pi/4$. Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(0, 0, 1, 1), \qquad g_1 = \frac{1}{\sqrt{2}}(-1, 1, 0, 0).$$

This is wavelets of Haar. Scaling function has the support on interval $[2, 3]$. Refinement equations: $\varphi(x) = \varphi(2x-2) + \varphi(2x-3)$ and $\psi(x) = -\psi(2x) + \psi(2x-1)$.

4.4. Parameter values $t = \pi/4$, $u = \pi/2$. Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(0, 1, 1, 0), \qquad g_1 = \frac{1}{\sqrt{2}}(0, 1, -1, 0).$$

This is wavelets of Haar. Scaling function has the support on interval $[1, 2]$. Refinement equations: $\varphi(x) = \varphi(2x-1) + \varphi(2x-2)$ and $\psi(x) = \psi(2x-1) - \psi(2x-2)$.

4.5. Parameter values $t = \pi/12$, $u = \pi/6$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3+\sqrt{3}, 1+\sqrt{3}, 1-\sqrt{3}, 3-\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-3+\sqrt{3}, 1-\sqrt{3}, -1-\sqrt{3}, 3+\sqrt{3}).$$

The result will be wavelets with coefficients which are obtained by permutation of coefficients of the classical Daubechies wavelets with the support of length 3. Refinement equations:

$$\varphi(x) = \frac{3+\sqrt{3}}{4}\varphi(2x) + \frac{1+\sqrt{3}}{4}\varphi(2x-1) + \frac{1-\sqrt{3}}{4}\varphi(2x-2) + \frac{3-\sqrt{3}}{4}\varphi(2x-3).$$

In figure 1 graphs of wavelets are shown.

4.6. Parameter values $t = 5\pi/12$, $u = -\pi/6$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3-\sqrt{3}, 1-\sqrt{3}, 1+\sqrt{3}, 3+\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-3-\sqrt{3}, 1+\sqrt{3}, -1+\sqrt{3}, 3-\sqrt{3}).$$

This example differs from previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from scaling function of example 4.5 with the using of argument replacement: $\varphi(3-x)$. It follows from the fact: if $\varphi(x)$ – scaling function with the compact support $[0, L]$ and the filter $\{h_n\}$ then function $\varphi(L-x)$ also is scaling with the filter $\{h_{L-n}\}$.The corresponding wavelet also can be obtained from previous as:

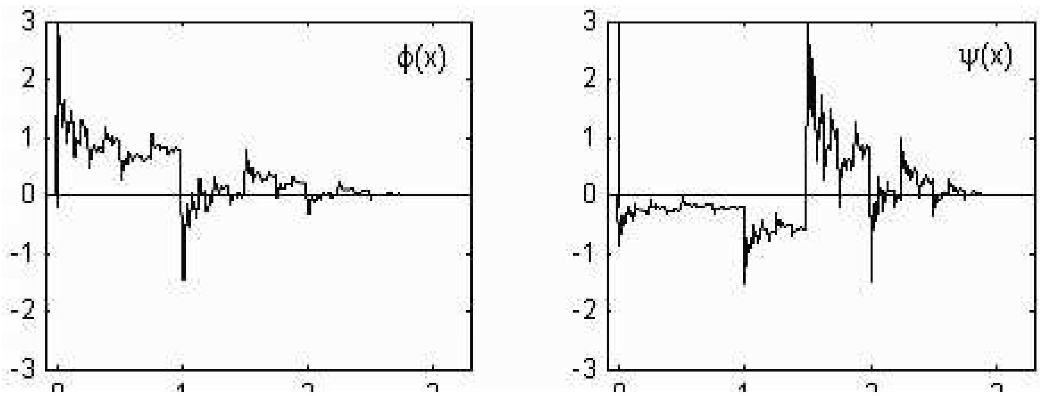

Figure 1: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = \pi/12$, $u = \pi/6$

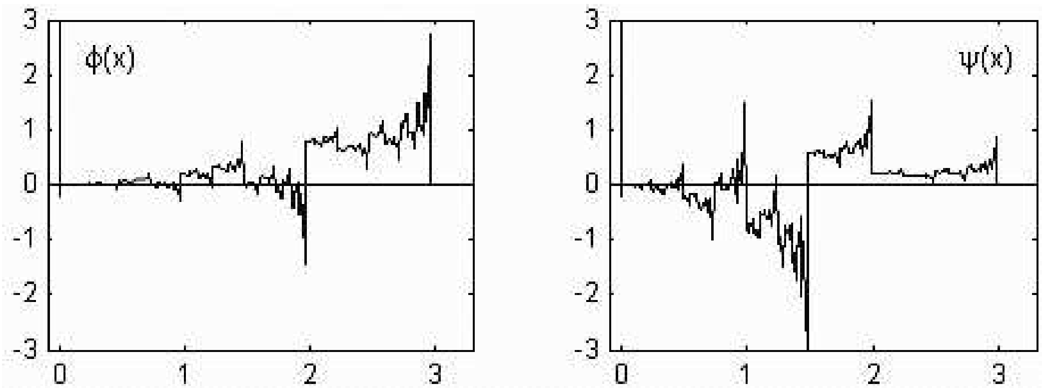

Figure 2: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = 5\pi/12$, $u = -\pi/6$

$-\psi(3-x)$. The graph of scaling function $\varphi(x)$ can be obtained from the graph of the Fig.1 by mirroring about the line $x = 3/2$. For the graph of the wavelet $\psi(x)$ it is necessary to add still mirroring about axis Ox (Fig. 2).

4.7. Parameter values $t = -\pi/12$, $u = \pi/3$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(1+\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-1+\sqrt{3}, 3-\sqrt{3}, -3-\sqrt{3}, 1+\sqrt{3}).$$

The result will be Daubechies wavelets with the support of length 3. Refinement equation:

$$\varphi(x) = \frac{1+\sqrt{3}}{4}\varphi(2x) + \frac{3+\sqrt{3}}{4}\varphi(2x-1) + \frac{3-\sqrt{3}}{4}\varphi(2x-2) + \frac{1-\sqrt{3}}{4}\varphi(2x-3).$$

In figure 3 graphs of wavelets are shown.

4.8. Parameter values $t = -5\pi/12$, $u = 2\pi/3$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(1-\sqrt{3}, 3-\sqrt{3}, 3+\sqrt{3}, 1+\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-1-\sqrt{3}, 3+\sqrt{3}, -3+\sqrt{3}, 1-\sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from Daubechies scaling function with the help of argument replacement: $\varphi(3-x)$, and wavelet is $-\psi(3-x)$.

4.9. Parameter values $t = \pi/6$, $u = \pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3}, 1+\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-1-\sqrt{3}, 1-\sqrt{3}, -3+\sqrt{3}, 3+\sqrt{3}).$$

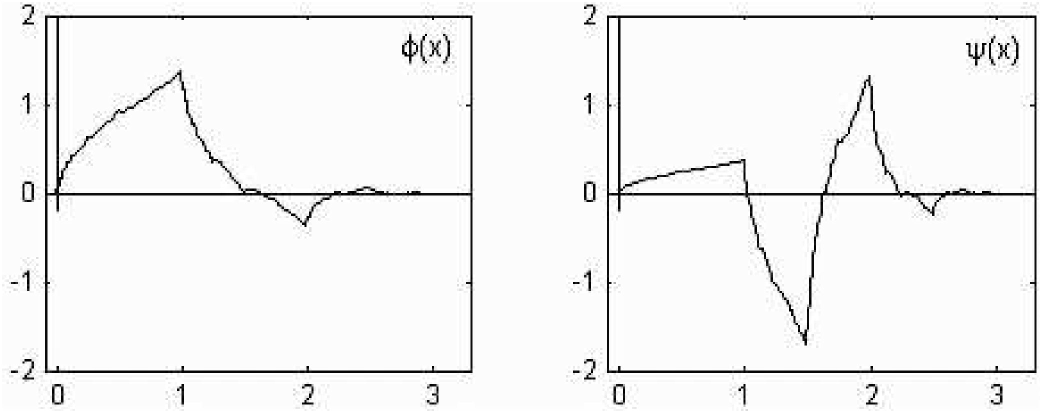

Figure 3: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t=-\pi/12$, $u=\pi/3$

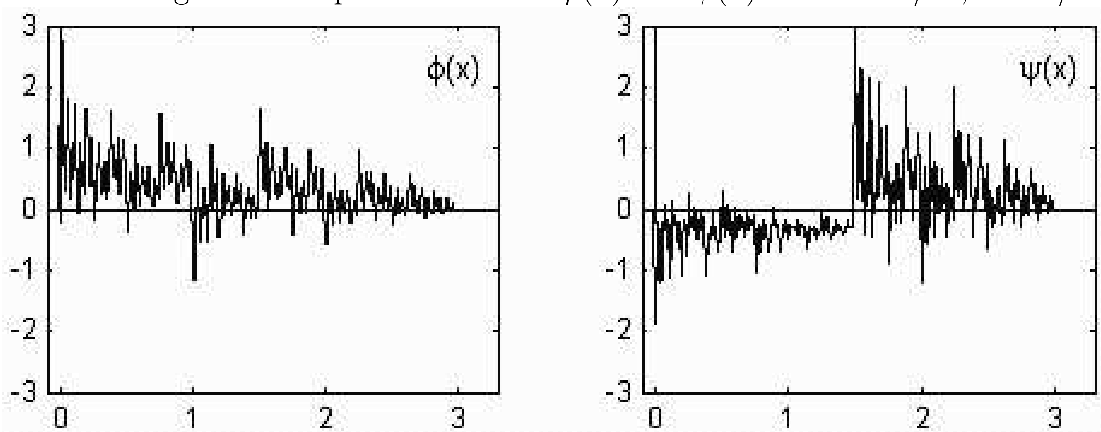

Figure 4: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t=\pi/6$, $u=\pi/12$

The result will be wavelets with coefficients which are obtained by permutation of Daubechies wavelets coefficients. The refinement equation:

$$\varphi(x)=\frac{3+\sqrt{3}}{4}\varphi(2x)+\frac{3-\sqrt{3}}{4}\varphi(2x-1)+\frac{1-\sqrt{3}}{4}\varphi(2x-2)+\frac{1+\sqrt{3}}{4}\varphi(2x-3).$$

In figure 4 graphs of wavelets are shown.

4.10. Parameter values $t=\pi/3$, $u=-\pi/12$. Coefficients of wavelets filters:

$$h_0=\frac{\sqrt{2}}{8}(1+\sqrt{3},1-\sqrt{3},3-\sqrt{3},3+\sqrt{3}),\qquad g_1=\frac{\sqrt{2}}{8}(-3-\sqrt{3},3-\sqrt{3},-1+\sqrt{3},1+\sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from the previous scaling function by replacement of argument: $\varphi(3-x)$, and wavelet is $-\psi(3-x)$.

4.11. Parameter values $t=-\pi/3$, $u=7\pi/12$. Coefficients of wavelets filters:

$$h_0=\frac{\sqrt{2}}{8}(1-\sqrt{3},1+\sqrt{3},3+\sqrt{3},3-\sqrt{3}),\qquad g_1=\frac{\sqrt{2}}{8}(-3+\sqrt{3},3+\sqrt{3},-1-\sqrt{3},1-\sqrt{3}).$$

The result will be wavelets with coefficients which are obtained by coefficients permutation of Daubechies wavelets with the support of length 3. Refinement equations:

$$\varphi(x)=\frac{1-\sqrt{3}}{4}\varphi(2x)+\frac{1+\sqrt{3}}{4}\varphi(2x-1)+\frac{3+\sqrt{3}}{4}\varphi(2x-2)+\frac{3-\sqrt{3}}{4}\varphi(2x-3).$$

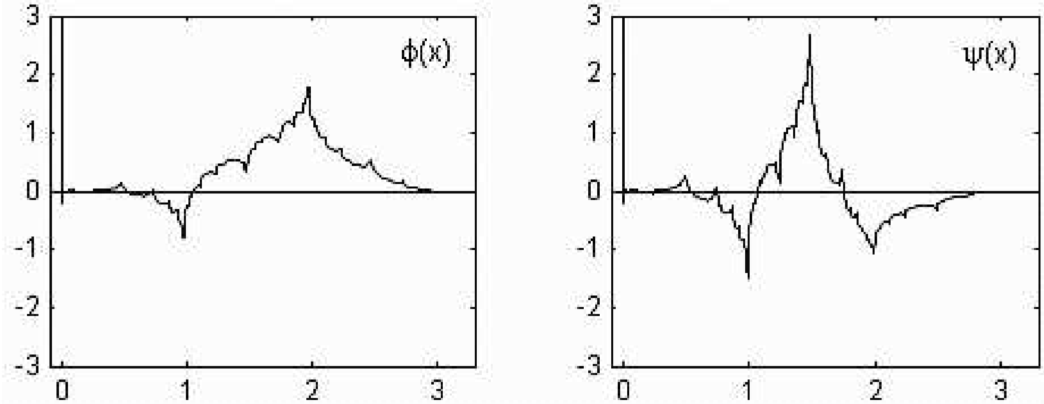

Figure 5: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = -\pi/3$, $u = 7\pi/12$

In figure 5 graphs of wavelets are shown.

4.12. Parameter values $t = -\pi/6$, $u = 5\pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3-\sqrt{3}, 3+\sqrt{3}, 1+\sqrt{3}, 1-\sqrt{3}), \qquad g_1 = \frac{\sqrt{2}}{8}(-1+\sqrt{3}, 1+\sqrt{3}, -3-\sqrt{3}, 3-\sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function and wavelet can be obtained from the previous by replacement of argument: $\varphi(3-x)$, $-\psi(3-x)$.

5. Construction of wavelets in case $N = 3$. In this section we shall show the scheme of scaling function and wavelets construction for $N = 3$. Though the diagonal matrix $D_k(w)$ can be anyone, we shall take for example the diagonal matrix $D_1(w) = \mathrm{diag}(1, w, 1)$, $|w| = 1$. The matrix A_0 is:

$$\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.$$

Elements of the second orthogonal matrix B_0 should satisfy to conditions:

$$b_{00} + b_{01} + b_{02} = 1, \quad b_{10} + b_{11} + b_{12} = 1, \quad b_{20} + b_{21} + b_{22} = 1.$$

The solution of this system will be any set of orthonormal vectors which coordinates satisfy to the equation of the plane $x_0+x_1+x_2 = 1$. It is obvious, that coordinates of basis vectors e_1, e_2, e_3 satisfy to this equation of plane. For this solution the matrix B_0 it is identity. And we obtain the wavelets of Haar,

$$A_0D_1(w)B_0 = \begin{pmatrix} 1/\sqrt{3} & w/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -w/\sqrt{2} & 0 \\ 1/\sqrt{6} & w/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}, \tag{23}$$

$$H(z) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} & z^3/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -z^3/\sqrt{2} & 0 \\ 1/\sqrt{6} & z^3/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ z & \rho z & \rho^2 z \\ z^2 & \rho^2 z^2 & \rho^4 z^2 \end{pmatrix},$$

$$H_0(z) = \frac{1}{3}(1 + z^2 + z^4), \qquad H_1(z) = \frac{1}{\sqrt{6}}(1 - z^4), \qquad H_2(z) = \frac{1}{3\sqrt{2}}(1 - 2z^2 + z^4).$$

The maximum degree of frequency function $H_0(z)$ is equal to four, the support length L is equal to two, as it is find from the formula $L(N-1)+1 = \deg(H_0(z)) + 1$.

It is easy to see, that scaling function $\varphi(x)$ is characteristic function of interval [0,2), $\varphi(x) = \chi_{[0,2)}(x)$. The refinement equation and wavelets (Fig. 6):

$$\varphi(x) = \varphi(3x) + \varphi(3x-2) + \varphi(3x-4),$$

$$\psi^1(x) = \frac{\sqrt{3}}{\sqrt{2}}\left(\varphi(3x) - \varphi(3x-4)\right),$$

$$\psi^2(x) = \frac{1}{\sqrt{2}}\left(\varphi(3x) - 2\varphi(3x-2) + \varphi(3x-4)\right),$$

Any other solution can be obtained by rotation of basis vectors e_0, e_1, e_2 in the plane $x_0 + x_1 + x_2 = 1$. We shall find these solutions. Let

$$M(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

– the matrix of rotations around of the axis e_0. Then

$$M_e(t) = A_0^{-1} M(t) A_0 = \frac{1}{3}\begin{pmatrix} 1+2\cos t & 1-\cos t+\sqrt{3}\sin t & 1-\cos t-\sqrt{3}\sin t \\ 1-\cos t-\sqrt{3}\sin t & 1+2\cos t & 1-\cos t+\sqrt{3}\sin t \\ 1-\cos t+\sqrt{3}\sin t & 1-\cos t-\sqrt{3}\sin t & 1+2\cos t \end{pmatrix}.$$

Let's make rotation $M_e(t)e_k$ of column vectors $e_0 = (1,0,0)$, $e_1 = (0,1,0)$, $e_2 = (0,0,1)$, and we obtain rows of the required matrix $B_0(t)$:

$$B_0(t) = \frac{1}{3}\begin{pmatrix} 1+2\cos t & 1-\cos t-\sqrt{3}\sin t & 1-\cos t+\sqrt{3}\sin t \\ 1-\cos t+\sqrt{3}\sin t & 1+2\cos t & 1-\cos t-\sqrt{3}\sin t \\ 1-\cos t-\sqrt{3}\sin t & 1-\cos t+\sqrt{3}\sin t & 1+2\cos t \end{pmatrix}. \qquad (24)$$

Then $H(t,w) = A_0 D_1(w) B_0(t) R(z)$ where the matrix $A_0 D_1(z^N)$ is represented by the formula (23), $B_0(t)$ – by the formula (24) and the matrix $R(z)$ is

$$R(z) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ z & \rho z & \rho^2 z \\ z^2 & \rho^2 z^2 & \rho^4 z^2 \end{pmatrix}.$$

Multiplying all these matrixes and choosing elements of the first column, we obtain,

$$H_0(t,z) = \frac{1}{9}\Big(2+\cos t-\sqrt{3}\sin t + (2-2\cos t)z + (2+\cos t+\sqrt{3}\sin t)z^2 +$$

$$+(1-\cos t+\sqrt{3}\sin t)z^3 + (1+2\cos t)z^4 + (1-\cos t-\sqrt{3}\sin t)z^5\Big), \qquad (25)$$

$$H_1(t,z) = \frac{1}{3\sqrt{6}}\Big(1+2\cos t + (1-\cos t-\sqrt{3}\sin t)z + (1-\cos t+\sqrt{3}\sin t)z^2 -$$

$$-(1-\cos t+\sqrt{3}\sin t)z^3 - (1+2\cos t)z^4 + (-1+\cos t+\sqrt{3}\sin t)z^5\Big), \qquad (26)$$

$$H_2(t,z) = \frac{1}{9\sqrt{2}}\Big(-1+4\cos t+2\sqrt{3}\sin t - (1-\cos t+3\sqrt{3}\sin t)z -$$

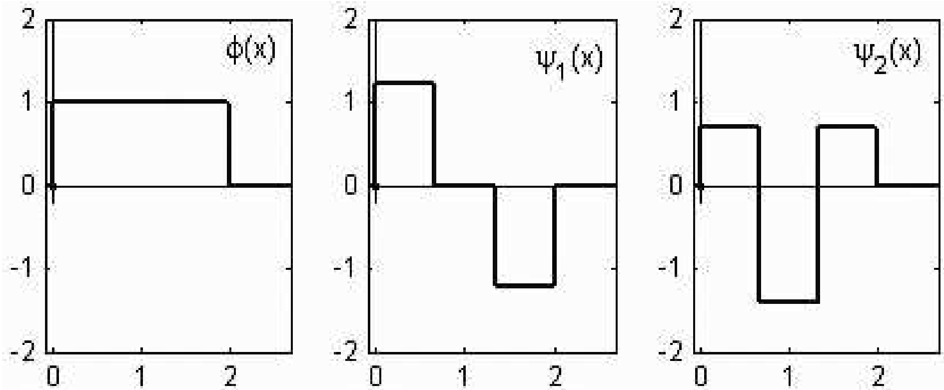

Figure 6: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t=0$

$$-(1+5\cos t+\sqrt{3}\sin t)z^2+$$
$$+(1-\cos t+\sqrt{3}\sin t)z^3+(1+2\cos t)z^4+(1-\cos t-\sqrt{3}\sin t)z^5\Big). \qquad (27)$$

6. Examples of scaling functions and wavelets for $N=3$. We shall calculate coefficients of the obtained frequency functions (25), (26) and (27) for various parameter values t. The obtained filters allow to find corresponding wavelets $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ by usual methods [5], [3]. It is enough to find scaling function $\varphi(x)$. Wavelets - functions $\psi^1(x)$ and $\psi^2(x)$ are defined by formulas

$$\psi^1(x)=\sqrt{N}\sum_{n\in\mathbb{Z}}g_n^1\varphi(Nx-n), \qquad \psi^2(x)=\sqrt{N}\sum_{n\in\mathbb{Z}}g_n^2\varphi(Nx-n)$$

with known filters $\{g_n^1\}$ and $\{g_n^2\}$ and function $\varphi(x)$.

Let's consider the following parameter values t: 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, π, $4\pi/3$. For each case graphs of wavelets-functions are shown.

6.1. Value of parameter $t=0$. This case has already been considered above. It is wavelets of Haar with the support $[0,2]$ (Fig. 6).

6.2. Value of parameter $t=\pi/6$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0=\frac{\sqrt{3}}{9}(2,2-\sqrt{3},2+\sqrt{3},1,1+\sqrt{3},1-\sqrt{3}),$$
$$g_1=\frac{\sqrt{6}}{18}(3+\sqrt{3},-3+\sqrt{3},\sqrt{3},-\sqrt{3},-3-\sqrt{3},3-\sqrt{3}),$$
$$g_2=\frac{\sqrt{6}}{18}(-1+3\sqrt{3},-1-\sqrt{3},-1-2\sqrt{3},1,1+\sqrt{3},1-\sqrt{3}).$$

The refinement equation:

$$\varphi(x)=\frac{1}{3}(2\varphi(3x)+(2-\sqrt{3})\varphi(3x-1)+(2+\sqrt{3})\varphi(3x-2)+\varphi(3x-3)+$$
$$+(1+\sqrt{3})\varphi(3x-4)+(1-\sqrt{3})\varphi(3x-5).$$

Graphs of wavelets are shown in figure 7

6.3. Value of parameter $t=\pi/4$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0=\frac{\sqrt{3}}{18}(4+\sqrt{2}-\sqrt{6},4-2\sqrt{2},4+\sqrt{2}+\sqrt{6},2-\sqrt{2}+\sqrt{6},2+2\sqrt{2},2-\sqrt{2}-\sqrt{6}),$$

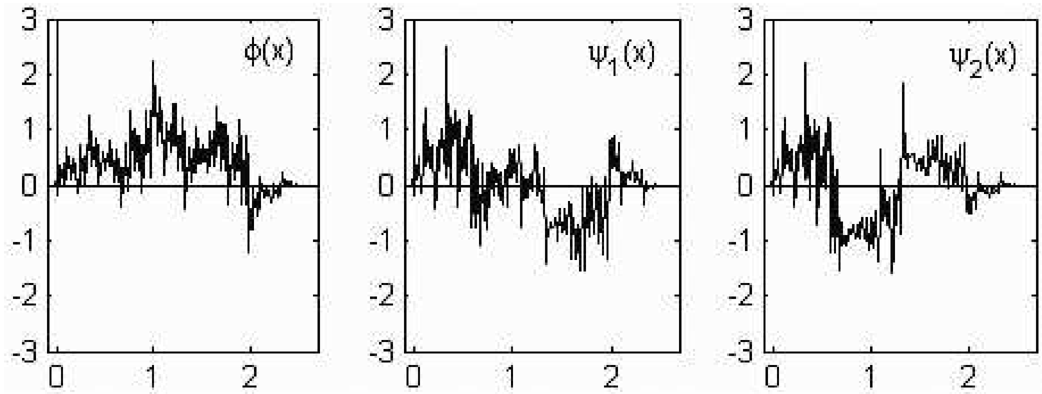

Figure 7: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/6$

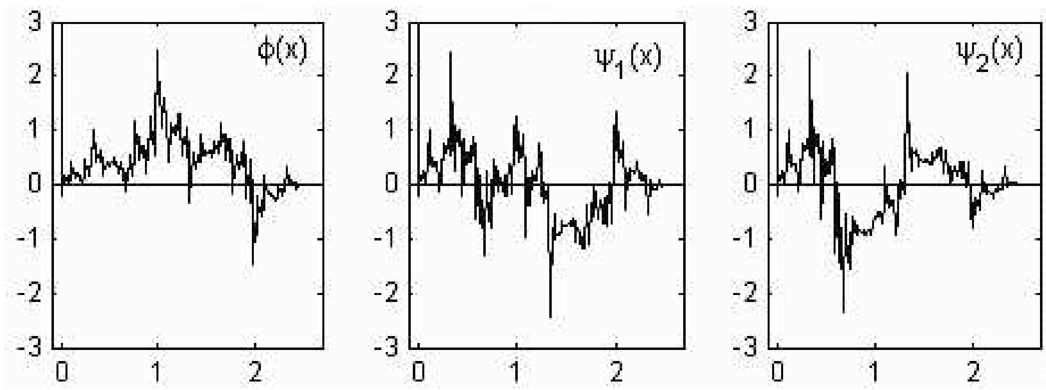

Figure 8: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/4$

$$g_1 = \frac{\sqrt{6}}{36}(2\sqrt{3}+2\sqrt{6}, -3\sqrt{2}+2\sqrt{3}-\sqrt{6}, 3\sqrt{2}+2\sqrt{3}-\sqrt{6}, -3\sqrt{2}-2\sqrt{3}+\sqrt{6},$$
$$-2\sqrt{3}-2\sqrt{6}, 3\sqrt{2}-2\sqrt{3}+\sqrt{6}),$$
$$g_2 = \frac{\sqrt{6}}{36}(-2+4\sqrt{2}+2\sqrt{6}, -2+\sqrt{2}-3\sqrt{6}, -2-5\sqrt{2}+\sqrt{6}, 2-\sqrt{2}+\sqrt{6},$$
$$2+2\sqrt{2}, 2-\sqrt{2}-\sqrt{6}).$$

Graphs of wavelets are shown in figure 8

6.4. Value of parameter $t = \pi/3$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$ are

$$h_0 = \frac{\sqrt{3}}{9}(1, 1, 4, 2, 2, -1),$$
$$g_1 = \frac{\sqrt{2}}{6}(2, -1, 2, -2, -2, 1), \qquad g_2 = \frac{\sqrt{6}}{18}(4, -5, -2, 2, 2, -1).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}(\varphi(3x) + \varphi(3x-1) + 4\varphi(3x-2) + 2\varphi(3x-3) + 2\varphi(3x-4) - \varphi(3x-5).$$

Graphs of wavelets are shown in figure 9.

6.5. Value of parameter $t = \pi/2$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

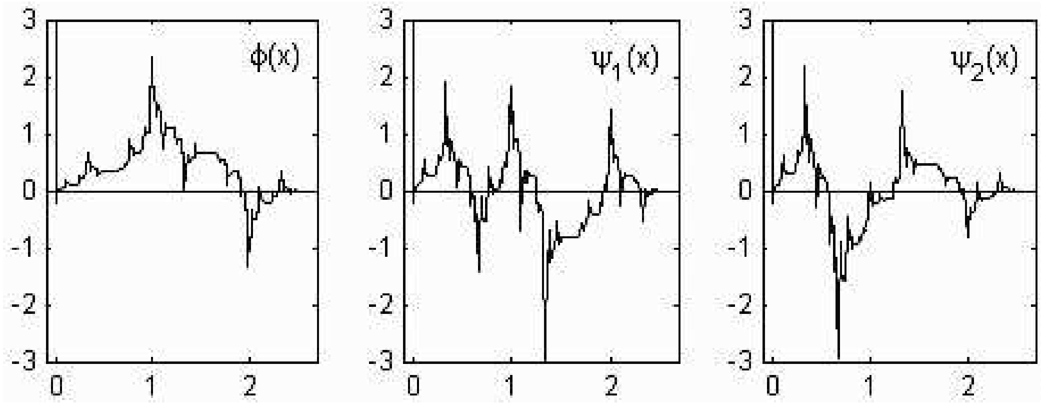

Figure 9: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/3$

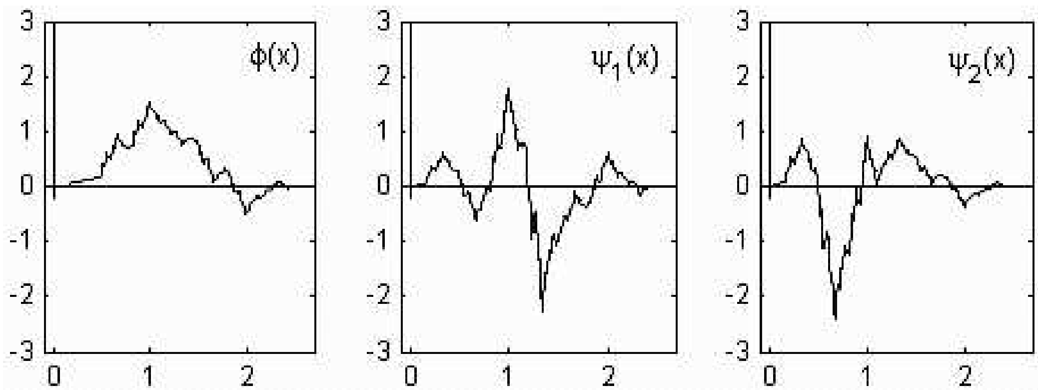

Figure 10: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/2$

$$h_0 = \frac{\sqrt{3}}{9}(2-\sqrt{3}, 2, 2+\sqrt{3}, 1+\sqrt{3}, 1, 1-\sqrt{3}),$$

$$g_1 = \frac{\sqrt{6}}{18}(\sqrt{3}, -3+\sqrt{3}, 3+\sqrt{3}, -3-\sqrt{3}, -\sqrt{3}, 3-\sqrt{3}),$$

$$g_2 = \frac{\sqrt{6}}{18}(-1+2\sqrt{3}, -1-3\sqrt{3}, -1+\sqrt{3}, 1+\sqrt{3}, 1, 1-\sqrt{3}).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}((2-\sqrt{3})\varphi(3x) + 2\varphi(3x-1) + (2+\sqrt{3})\varphi(3x-2) + (1+\sqrt{3})\varphi(3x-3)+$$

$$+\varphi(3x-4) + (1-\sqrt{3})\varphi(3x-5).$$

Graphs of wavelets are shown in figure 10.

6.6. Value of parameter $t = 2\pi/3$**.** Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$ is

$$h_0 = \frac{1}{\sqrt{3}}(0, 1, 1, 1, 0, 0),$$

$$g_1 = \frac{1}{\sqrt{2}}(0, 0, 1, -1, 0, 0), \qquad g_2 = \frac{1}{\sqrt{6}}(0, -2, 1, 1, 0, 0).$$

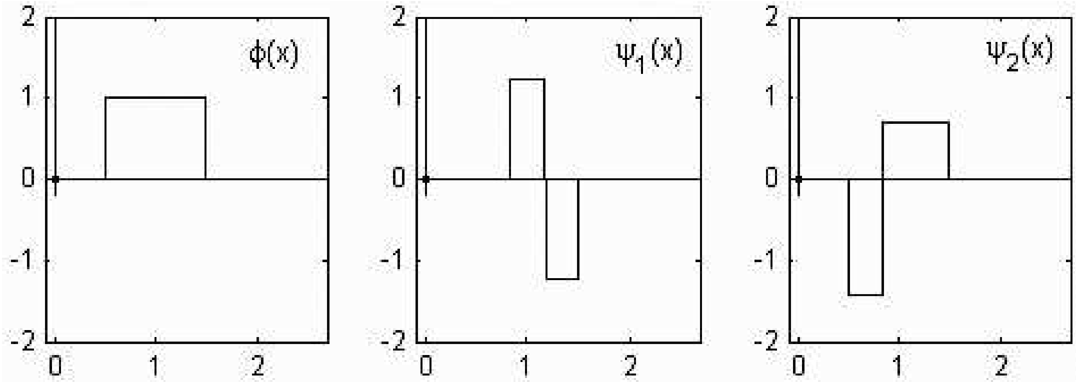

Figure 11: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = 2\pi/3$

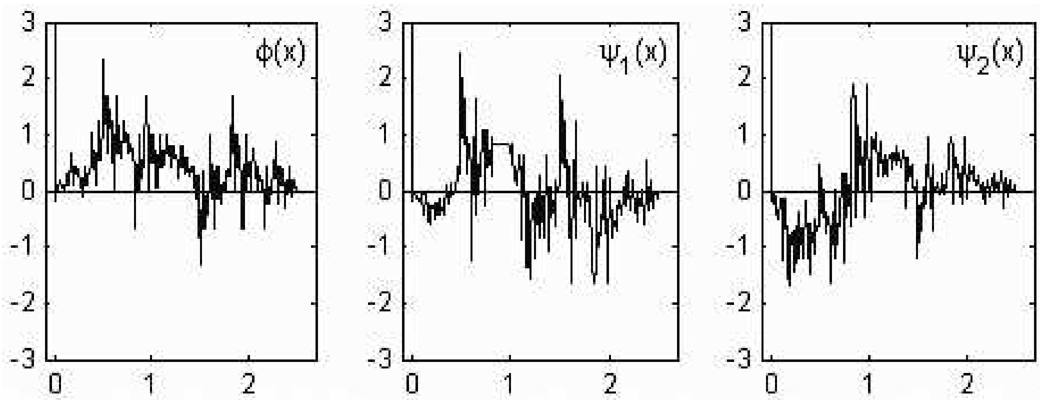

Figure 12: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi$

It is wavelets of Haar. The scaling function $\varphi(x)$ is characteristic function of interval $[1/2, 3/2)$, $\varphi(x) = \chi_{[1/2,3/2)}(x)$. The refinement equation and wavelets:

$$\varphi(x) = \varphi(3x-1) + \varphi(3x-2) + \varphi(3x-3),$$

$$\psi^1(x) = \frac{\sqrt{3}}{\sqrt{2}}(\varphi(3x-2) - \varphi(3x-3)),$$

$$\psi^2(x) = \frac{1}{\sqrt{2}}(-2\varphi(3x-1) + \varphi(3x-2) + \varphi(3x-3).$$

In figure 11 graphs of wavelets are shown.

6.7. Value of parameter $t = \pi$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0 = \frac{\sqrt{3}}{9}(1, 4, 1, 2, -1, 2),$$

$$g_1 = \frac{\sqrt{2}}{6}(-1, 2, 2, -2, 1, -2), \qquad g_2 = \frac{\sqrt{6}}{18}(-5, -2, 4, 2, -1, 2).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}(\varphi(3x) + 4\varphi(3x-1) + \varphi(3x-2) + 2\varphi(3x-3) - \varphi(3x-4) + 2\varphi(3x-5).$$

Graphs of wavelets are shown in figure 12.

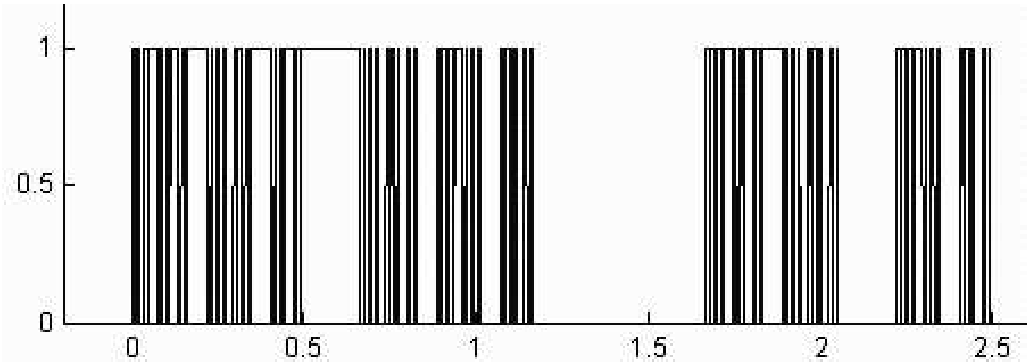

Figure 13: Graph of function $\varphi(x)$ for $t = 4\pi/3$

6.8. Value of parameter $t = 4\pi/3$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0 = \frac{1}{\sqrt{3}}(1, 1, 0, 0, 0, 1),$$

$$g_1 = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1), \qquad g_2 = \frac{1}{\sqrt{6}}(-2, 1, 0, 0, 0, 1).$$

The refinement equation and frequency functions:

$$\varphi(x) = \varphi(3x) + \varphi(3x - 1) + \varphi(3x - 5),$$

$$H_0(z) = \frac{1}{3}(1 + z + z^5), \qquad H_1(z) = \frac{1}{\sqrt{6}}(z - z^5), \qquad H_2(z) = \frac{1}{3\sqrt{2}}(-2 + z + z^5).$$

Let's mark, that scaling function $\varphi(x)$ has a complicated structure. Its support has fractal properties.

References

[1] O. Bratteli, D. E. Evans, and P. E. T. Jorgensen, Compactly supported wavelets and representations of the Cuntz relations. Appl. Comput. Harmon. Anal. Vol. 8, 2000, 166-196. arXiv.org:math.FA/9912129

[2] O. Bratelli, P. E. T. Jorgensen, Wavelet filters and infinite-dimensional unitary groups. arXiv.org:math.FA/0001171 v3, (2000), 31 p.

[3] I. Daubechies, Ten Lectures on Wavelets. CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[4] P. N. Podkur, N. K. Smolentsev, Construction of some types wavelets with coefficient of scaling N. arXiv.org:math.FA/0612573, 2006, 19 P.

[5] N. K. Smolentsev, Osnovy teorii weivletov. Weivlety v MATLAB. M., DMK Press, 2005 (in Russian).